UNIT 5 : INFINITE SERIES II

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5.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- different tests of convergence
- alternating series, Leibniz's test
- absolute convergence and conditional convergence.

5.2 INTRODUCTION

In this unit, we will discuss some special tests of convergence. These tests will help us to determine the convergence of a wide variety of series. Furthermore, we will discuss alternating series, and Leibniz's test

for alternating series. Also, we will discuss absolute convergence and conditional convergence.

5.3 DIFFERENT TESTS OF CONVERGENCE

Here we will discuss eight speical tests of convergence. It will help us to determine the convergence of a wide variety of series. We will state and prove D'Alembert's Ratio Test, Cauchy's Root Test, Raabe's Test, Logarithmic Test, Cauchy's Condensation Test, De Morgan and Bertrand's Test. We will state Gauss's Test and Cauchy's Integral Test without proof.

5.3.1 D' Alembert's Ratio Test

(after the name of the French mathematician Jean le Roud d'Alembert (1717-1783) Theorem 5.1

Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms such that $\lim_{n \to \infty} \frac{u_n + 1}{u_n} = L$

I. If L < 1, then $\sum_{n=1}^{\infty} u_n$ converges

II. If L > 1, them $\sum_{n=1}^{\infty} u_n$ diverges

III. If L = 1, then test fails.

Proof I. Let L < 1.

We choose a positive number ε such that L + ε < 1.

$\because \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = L$, there exists a positive integer m such that

$L - \varepsilon < \frac{u_{n+1}}{u_n} < L + \varepsilon$, for all $n \geq m$.

Let L + ε = r

Then 0 < r < 1

We have $\frac{u_{m+1}}{u_m} < r$,

$\frac{u_{m+2}}{u_{m+1}} < r$,

.....

$$\frac{u_n}{u_{n-1}} < r, \qquad n > m$$

$$\therefore \frac{u_{m+1}}{u_m}.\frac{u_{m+2}}{u_{m+1}}.....\frac{u_n}{u_{n-1}} < r,\ r.....r \quad \{(n-m) \text{ times}\}$$

or, $$\frac{u_n}{u_m} < r^{n-m}, \qquad n > m.$$

or, $$u_n < u_m.r^n.r^{-m}$$

or, $$u_n < \frac{u_m}{r^m} r^n$$

Let, $$\frac{u_m}{r^m} = k > 0$$

$$\therefore u_n < k\ r^n$$

Now $\sum r^n$ is a geometric series with common ratio r, where 0 < r < q.

$\therefore \sum r^n$ is convergent

$\therefore \sum r^n$ is also convergent (Thm 4.5 I (i))

II. Let L > 1

We choose a positive number ε such that $L - \varepsilon > 1$.

$\because \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = L$, there exist a positive integer t such that

$$L - \varepsilon < \frac{u_{n+1}}{u_n} < L + \varepsilon, \text{ for all } n \geq m$$

Let $L - \varepsilon = p$. Then p > 1

We have $$\frac{u_{t+1}}{u_t} > p$$

$$\frac{u_{t+2}}{u_{t+1}} > p$$

$$\frac{u_n}{u_{n-1}} > p, \qquad n > t$$

$$\therefore \frac{u_{t+1}}{u_t}.\frac{u_{t+2}}{u_{t+1}}.....\frac{u_n}{u_{n-1}} < p,\ p.....p, \qquad n > t$$

or $\quad \frac{u_n}{u_t} > p^{n-t}, \quad$ for all $n > t$

or $\quad u_n > \frac{u_t}{p^t}\ p^n, \qquad n > t$

Let $\quad \frac{u_t}{p^t} = K > 0$

$\therefore u_n < K\ p^n, \qquad n > t$

Now $\sum p^n$ is a geometric series nth common ration p, where p > 1.

$\therefore \sum p^n$ is divergent.

$\therefore \sum p^n$ is also divergent (Thm 4.5 I)

III. If L = 1, no conclusion can be drawn. In some cases, the series may converge. In some cases, it may diverge.

Let us consider the series $\sum \frac{1}{n}$

Here $u_n = \frac{1}{n}, \quad u_{n+1} = \frac{1}{n+1}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

We know that the series $\sum \frac{1}{n}$ divergent (harmonic series)

Again, let us consider the series $\sum \frac{1}{n^2}$

Here $u_n = \frac{1}{n^2}, \quad u_{n+1} = \frac{1}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2} = 1$$

We know that the series is convergent (p-series with p = 2)

Example 1. Examine the convergence of the series

$$\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x4}{4}+.....\infty, \qquad x > 0$$

Solution : Here $u_n=\frac{x^n}{n},\ u_{n+1}=\frac{x^{n+1}}{n+1}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{u^{n+1}}{n+1}.\frac{n}{x^n}$$

$$=\lim_{n\to\infty}\frac{n}{n+1}.x=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\,x$$

$= x$

$\therefore$ by d' Alembert's ratio that, the series is convergent if x > 1, and divergent if x < 1.

If x = 1, the series becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....\infty$

which is convergent.

5.3.2 Cauchy's Root Test

(after the name of the French mathematician Augustin-Louis Cauchy (1789-1857).

Theorem 5.2. Let $\sum_{n=1}^{\infty}u_n$ be a series of positive terms such that

$$\lim_{n\to\infty}u_n^{\frac{1}{n}}=L$$

I. If L < 1, then $\sum_{n=1}^{\infty}u_n$ converges

II. If L > 1, then $\sum_{n=1}^{\infty}u_n$ diverges

III. If L = 1, then the test fails.

Proof : I Let L < 1.

We choose a positive number ε such that L + ε < 1

there exists a positive integer m such that

$L-\varepsilon \; u_n^{\frac{1}{n}} = L < L + \varepsilon$, for all $n \geq m$

or, $(L-\varepsilon)^n < u_n < (L+\varepsilon)^n$, for all $n \geq m$

Let $L + \varepsilon = r$

Then $0 < r < 1$

And $u_n < r^n$, for all $n \geq m$

Now $\sum r^n$ is a geometric series with common ratio r, $0 < r < 1$.

$\therefore \sum r^n$ is convergent

$\therefore \sum u^n$ is also convergent (Thm 4.5 I)

II. Let $L > 1$.

We choose a positive number ε such that $L - \varepsilon > 1$.

$\therefore \lim_{n \to \infty} u_n^{\frac{1}{n}} = L$, there exists a positive integer t such that

$L-<\varepsilon \; u_n^{\frac{1}{n}} < L+\varepsilon$, for all $n \geq t$

or, $(L-\varepsilon)^n < u_n < (L+\varepsilon)^n$, for all $n \geq t$

Let $L - \varepsilon = p$, Then $p > 1$

Also $P^n < u_n$, i.e. $u_n > p^n$, for all $n \geq t$

Now $\sum p^n$ is a geomtric series with common ratio p, where $p > 1$.

$\therefore \sum p^n$ is divergent.

$\therefore \sum u^n$ is also divergent (Thm 4.5I)

III. If L = 1, no conclusion can be drawn

In some cases, the series may converge. In some cases, it may diverge.

Let us consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$

In case f the cases $\lim_{n \to \infty} u_n^{\frac{1}{n}} = 1$

Now $\sum \frac{1}{n}$ is divergent, whereas the series $\sum \frac{1}{n^2}$ is convergent.

Example 2 Examine the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

Solution : Here $u_n = \frac{1}{(\log n)^n}$

$$\therefore \quad u_n^{\frac{1}{n}} = \left(\frac{1}{(\log n)^n}\right)^{\frac{1}{n}} = \frac{1}{\log n}$$

$$\therefore \quad \lim_{n \to \infty} u_n^{\frac{1}{n}} = 0 < 1$$

$\therefore \quad \sum_{n=2}^{\infty} u_n$ is convergent (Thm 5.2)

5.3.3 Raabe's Test

(after the name of the Swiss mathematician Joseph Luding Raabe (1801-1859)

Theorem 5.3 Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms, and let

$$\lim_{n \to \infty} \left(n \left(\frac{u_n}{u_{n+1}} - 1\right)\right) = L$$

Then the series converges or diverges according as L > 1 or L < 1.

Proof : Let $\upsilon_n = \frac{1}{n^p}$, where p > 1

Clearly $\sum \upsilon_n$ is convergent. (p-series, p > 1)

$\therefore \quad \sum u_n$ is also convergent if

$$\frac{u_n}{u_{n+1}} > \frac{\upsilon_n}{\upsilon_{n+1}}$$

i.e., if $\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1 + \frac{1}{n}\right)^p$

i.e., if $\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots\ldots\infty$

i.e., if $n\left(\frac{u_n}{u_{n+1}}-1\right) > p+\frac{p(p+1)}{2!}.\frac{1}{n}+.....\infty$

i.e., if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) > p > 1$

$\therefore$ The series $\sum u_n$ is convergent if L > 1.

$\sum u_n$ is divergent if L < 1.

In this case, we take $\upsilon_n = \frac{1}{n^p}$ where p < 1.

Note : Raabe's test is applied, when ratio test fails.

If L =1, then Raabe's test fails.

Example 3: Examine the convergence of the series

$$1+\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}\frac{1}{5}+\frac{1.3.5}{2.4.6}\frac{1}{7}+.....\infty$$

Solution: Here $u_n = \frac{1.3.5.....(2n-3)}{2.4.6.....(2n-2)}\ \frac{1}{2n-1}$, for all $n \geq 2$

$$u_{n+1} = \frac{1.3.5.....(2n-3).(2n-1)}{2.4.6.....(2n-2).2n}\ \frac{1}{2n-1}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(2n-1)^2}{2n(n+1)},\ n \geq 2$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$$

$\therefore$ d' Alembert's ratio test fails.

Now let us apply Raabe's test

Now $n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n\ (2n+1)}{(2n-1)^2}-1\right)$

$$= n\frac{(4n^2+2n-(2n-1)^2)}{(2n-1)^2}$$

$$= n.\frac{(4n^2+2n-4n^2+4n-1)}{(2n-1)^2}$$

$$= n\,\frac{(6n-1)}{(2n-1)^2}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} \frac{n(6n-1)}{(2n-1)^2}$$

$$= \lim_{n\to\infty} \frac{n\,n\,(6-\frac{1}{n})}{n^2\,(2-\frac{1}{n})^2}$$

$$= \frac{3}{2} > 1$$

$\therefore \sum u_n$ is convergent (Thm 5.3)

5.3.4 Logarithmic Test

Theorem 5.4 Let $\sum u_n$ be a series of positive terms, and let

$$\lim_{n\to\infty} (n \log \frac{u_n}{u_{n+1}}) = L$$

Then the series converges or diverges according as L > 1 or L < 1.

Proof : Let L> 1

We take a number p such that L > P > 1

Let us consider the series $\sum \upsilon_n$, where $\upsilon_n = \frac{1}{n^p}$

Now, $\sum \upsilon_n$ is convergent (p-series, p > 1)

$\sum u_n$ will be convergent if, after some particular terms,

$$\frac{\upsilon_n}{\upsilon_{n+1}} = \frac{1}{n^p}\,\frac{(n+1)^p}{1}$$

$$= \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p$$

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$$

i.e., if $\log \frac{u_n}{u_{n+1}} > \log \left(1+\frac{1}{n}\right)^p$

i.e., if $\log \frac{u_n}{u_{n+1}} > p \log \left(1+\frac{1}{n}\right)$

i.e., if $\log \frac{u_n}{u_{n+1}} > p \left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}.....\infty\right)$

i.e., if $n \log \frac{u_n}{u_{n+1}} > p \left(1-\frac{1}{2n}+\frac{1}{3n^2}.....\infty\right)$

i.e., if $n \log \frac{u_n}{u_{n+1}} > p+\frac{p}{n}\left(-\frac{1}{2}+\frac{1}{3n}-.....\infty\right)$

For large values of n, the left hand side tends to L, whereas the right hand side tends to p. But L > p. Hence the inequality is satisfied for sufficiently large values of n.

$\therefore$ The given series is convergent, if L > 1.

Similarly it can be proved that the series is divergent if L < 1.

Note: 1. When the d' Alembert's ratio test fails, the above test is applied. Moreover if n occurs as an exponent in $\frac{u_n}{u_{n+1}}$, this test should be applied.

Furthermore, if we take logarithem of $\frac{u_n}{u_{n+1}}$, the evaluation of limit may become easier in some cases.

2. No conclusion can be made when L = 1.

If $\lim_{n\to\infty}\left[(n\log\frac{u_n}{u_{n+1}}-1)\log n\right]=p$, then the series $\sum u_n$ converges or diverges according as p > 1 or p < 1.

Theorem 5.5

5.3.5 Cauchy's Condensation Test

Let {f(n)} be a monotonic decreasing sequence of positive real numbers and a be a positive integer greater than 1.

Then the series

f (1) + f (2) + f (3) + ...+ f (n) + ...

and a f (a) + a^2 f (a^2) + a^3 f (a^3) + ...+ a^n f (a^n) + ... converge or diverge together.

Proof : The series can be written as

$$\left(f(1)+f(2)+.....+f(a)\right)+\left(f(a+1)+f(a+2)+.....f(a^2)\right)$$

$$+\left(f(a^2+1)\right)+\left(f(a^2+2)+.....+f(a^3)\right)+.....+\left(f(a^m+1)+f(a^m+2)+.....+f(a^{m+1})\right)+.....$$

The terms of the (m + 1) the group are

$$f(a^m+1)+f(a^m+2)+.....+f(a^{m+1})$$

Number of terms in this group

$$= a^{m+1}-a^m = a^m(a-1)$$

$\because$ { f (n)} is a monotonic decreasing sequence, f (a^{m+1}) is the smallest term in this group. Therefore,

$$f(a^m+1)+f(a^m+2)+.....+f(a^{m+1})$$

$$>a^m(a-1\ f(a^{m+1})=\frac{a-1}{a}\ a^{m+1}\ f(a^{m+1})$$

Putting m = 0, 1, 2, ... successively in this inequality, we get

$$f(2)+f(3)+.....+f(a)>\frac{a-1}{a}\ af(a)$$

$$f(a+1)+f(a+2)+.....+f(a^2)>\frac{a-1}{a}\ a^2 f(a^2)$$

.....

.....

Adding, we get

$$\sum f(n)-f(1)>\frac{a-1}{a}\ \sum a^n\ f(a^n)$$

i.e. $$\frac{a-1}{a}\ \sum a^n\ f(a^n)<\sum f(n)-f(1) \qquad (1)$$

From (1), we see that if the series is divergent, then is also divergent.

Again, since {f (n)} is a monotonic decreasing sequence, each term in the (m+1) the group is less than f (a^m). Therefore

f (a^m +1) + f (a^m +2) + ...+ f (a^{m+1}) < a^m (a – 1) f (a^m)

Putting m = 0, 1, 2, ... successively,

We get

f (2) + f (3) + ... + f (a) < (a – 1) f (1)

f (a + 1) + f (a + 2) + ... f (a^2) < a (a –1) f (a)

.....

.....

.....

Adding, we get

$$\sum f(n) - f(1) > \frac{a-1}{a} \sum a^n f(a^n) + (a-1) f(1) \qquad (2)$$

From (2), we see that if the series $\sum a^n f(a^n)$ is convergent, then $\sum f(n)$ is also convergent.

Cololllary The series Let $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$, (p > 0) is convergent when p > 1, and divergent when $0 < p \le 1$

Let $f(n) = \frac{1}{n(\log n)^p}, \qquad n \ge 2$

Now {log n} is an increasing sequence, and p > 0. Therefore

$$(\log (n+1))^p > (\log n)^p$$

$$\therefore (n+1) (\log (n+1)^p > n (\log n)^p$$

$\therefore \{f(n)\}_{n=2}^{\infty}$ is a monotonic decreasing sequence.

$\therefore$ by Cauchy's condensation test, the series $\sum_{n=1}^{\infty} f(n)$ converges or diverges for the same value of p as the series

$$\sum \frac{a^n}{a^n (\log a^n)^p}, \qquad a > 1$$

i.e. $\sum \frac{1}{n (\log a)^p}$

i.e. $\sum \frac{1}{n^p (\log a)^p}$

Now $\frac{1}{(\log a)^p}$ is a constant, and the series $\sum \frac{1}{n^p}$ converges or

diverges according as p > 1 or p $\leq$ 1.

$\therefore$ The given series is convergent if p > 1, and divergent if $0 < p \leq 1$

5.3.6 De Morgan and Bertrand's Test

(after the names of Augustus De Morgan (1806-1871) and Joseph Bertrand (1822-1900)

Theorem 5.6 Let $\sum u_n$ be a series of positive terms, and let

$$\lim_{n \to \infty} \left[\log n \left(n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right) \right] = L$$

Then the series converges or diverges according as L > 1 or L < 1.

Proof : Let L > 1

We choose a number p such that L > p > 1.

Let us consider the series $\sum \upsilon_n$, where

$$\upsilon_n = \frac{1}{n\ (\log n)^p}, \quad n \geq 2, \quad p > 0$$

Now the series $\sum \upsilon_n$ is convergent when p > 1.

Therefore $\sum u_n$ is convergent, if after some particular terms

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)\ (\log (n+1))^p}{n\ (\log n)^p}$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right) \left(\frac{\log n + \log (1 + \frac{1}{a})}{n\ (\log n)^p} \right)^p$$

i.e. $$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right) \left(1 + \frac{p}{\log n} \left(\frac{1}{n} - \frac{1}{2n^2} + \ldots\ldots\infty \right)\right)$$

i.e. $$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} \frac{p}{\log n} + \ldots\ldots\ldots\ldots$$

i.e. $$n \left(\frac{u_n}{u_{n+1}} - 1 \right) > 1 + \frac{p}{\log n} + \ldots\ldots\ldots$$

i.e. $\log n\left(\left(n\frac{u_n}{u_{n+1}}-1\right)-1\right) > p+$ terms involving nor log n in the denominator.

Now, as n becomes indefinitely large, the left hand side and the right hand side of the above inquality tend to L and p respectively. But L > p. Therefore, for large values of n, the inequality is satisfied. Therefore $\sum u_n$ is convergent.

Similarly, it can be proved that the series is divergent of L <1.

5.3.7 Gauss's Test

(after the name of the German mathematician Carl Friedrich Gauss (1777-1855). We state this test without proof. It is a very strong test. It includes both D'Alembert's Ratio Test and Raabe's Test.

Theorem 5.7 Let $\sum u_n$ be a series of positive terms.

Let $\frac{u_n}{u_{n+1}} = a+\frac{b}{n}+\frac{r_n}{n^p}$

where a, b, p are real numbers, a > 0, p > 1, and $\{r_n\}$ is a bounded sequence.

Then

(i) $\sum u_n$ converges, if a > 1

(ii) $\sum u_n$ diverges, if a < 1

(iii) $\sum u_n$ converges, if a = 1, b >1

(iv) $\sum u_n$ diverges, if a = 1, b $\leq$ 1

5.3.8 Cauchy's Integral Test

[We state the theorem without proof].

Theorem 5.8 let f be a real - valued function with domain [1, ∞[

Let

(i) $f(x) \geq 0, \ \forall x \geq 1$

(ii) $x < y \Rightarrow f(x) \geq f(y)$

i.e. f is a nonnegative monotonically decreasing function.

(iii) The function f induce the following infinite series

f (1) + f (2) + f (3) + ... + f (n) + ...

Then the series $\sum_{n=1}^{\infty} f(x)$ and the infinite integral $\int_{1}^{\infty} f(x)\ dx$ converge or diverge together.

5.4 ALTERNATING SERIES

An infinite series $\sum u_n$ is called an alternating series if any two consecutive terms of the series are of opposite signs.

For example $1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....\infty$ is an alternating series .

5.4.1 Leibniz's test

(after the name of the German mathematician Gottfried Wilhelm Leibuiz (1646-1716)

Theorem 5.9 Let $\{u_n\}$ be a monotonic decreasing sequence of positive real numbers, and let $\lim_{n \to \infty} u_n = 0$. Then the alternating series

$u_1 - u_2 + u_3 - u_4 + ...\infty$ is convergent.

Proof : Let $S_n = u_1 - u_2 + u_3 + u_4 ..... + (-1)^{n+1} u_n$

Then $S_{2n} = u_1 - u_2 + u_3 + u_4 + ...u_{2n} + u_{2n+1} - u_{2n+2}$

$\therefore S_{2n} - S_{2n} = u_{2n+1} - u_{2n+2}$

$\geq 0, \quad \forall\ n \in N$

$\therefore$ The sequence $\{S_{2n}\}$ is a monotonic increasing sequence.

Again, $S_{2n-1} = u_1 - u_2 + u_3 - u_4 + ...u_{2n-1}$

$S_{2n+1} = u_1 - u_2 + u_3 - u_4 + ...u_{2n-1} - u_{2n} + u_{2n+1}$

$\therefore\ S_{2n+1} - S_{2n-1} = -u_{2n} + u_{2n+1}$

$\leq 0, \quad \forall n \in N$

$\therefore$ The sequence $\{S_{2n+1}\}$ is a monotonic decreasing sequence.

Now $S_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ldots\ldots - u_{2n}$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots\ldots\ldots\ldots - u_{2n}$

$< u_1$

$\therefore$ The sequence $\{S_{2n}\}$ is bounded above.

Also, $S_{2n+1} = u_1 - u_2 + u_3 + u_4 + \ldots\ldots\ldots - u_{2n+1}$

$= (u_1 - u_2) + (u_3 - u_4) + \ldots\ldots\ldots\ldots - u_{2n+1}$

$> u_1 - u_2$

$\therefore$ The sequence $\{S_{2n+1}\}$ is bounded below.

$\therefore$ The sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ are convergent.

Now $\lim_{n \to \infty} (S_{2n+1} - S_{2n})$

$= \lim_{n \to \infty} u_{2n+1} = 0$

$\therefore$ both the sequence $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the same limit.

$\therefore$ The sequence $\{S_n\}$ is convergent.

$\therefore$ The series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ldots\ldots\infty$ is convergent.

Example 4. Let us consider the series

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots\infty$$

Here each $u_i > 0$

The sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\right\}$ is monotonic decreasing.

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

$\therefore$ The series $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots\infty$ is convergent. (by Leibniz's test)

5.5 ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

Let $\sum_{n=1}^{\infty} u_n$ be a series of real numbers.

I. If the series $\sum_{n=1}^{\infty} |u_n|$ converges, then the series $\sum u_n$ is said to be absolutely convergent.

II. If $\sum u_n$ converges, but $\sum |u_n|$ diverges, then the series $\sum u_n$ is said to be conditionally convergent.

Example 5 : Let us consider the series

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}\ .....$$

Here each $u_i > 0$

The sequence $\left\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}\ .....\right\}$ is monotonic decreasing.

Now $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{3^{n-1}} = 0$

$\therefore$ The series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}\ .....\infty$ is convergent. (by Leibniz's test)

Now let us consider the series $\sum |u_n|$, i.e.

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}\ .....\infty$$

clearly it is a geometric series with common ratio $\frac{1}{3}$.

$\therefore$ The series is convergent.

Hence the give series is absolutely convergent.

Remark : We will prove that if an infinite series is absolutely convergent, then it is convergent.

Example 6 : Let us consider the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}.....$

We have seen that the above series is convergent (Example 4)

Now let us consider the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+.....\infty$

We have seen that the above series is divergent (4.5 II Harmonic Series)

$\therefore$ The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+.....\infty$ is conditionally convergent.

Theorem 5.10 An absolutely convergent series is convergent.

Proof : Let the series $\sum u_n$ be absolutely convergent. Then the series $\sum |u_n|$ is convergent.

Let $\{S_n\}$ be the sequence of partial sums of $\sum u_n$. Then

$S_n = u_1 + u_2 + u_3 + ... + u_n$

Let $\{T_n\}$ be the sequence of partial sums of $\sum |u_n|$.

Now, $\sum |u_n|$ is convergent.

$\therefore$ The sequence $\{T_n\}$ is convergent.

$\therefore$ $\{T_n\}$ is Cauchy sequence.

$\therefore$ for any $\varepsilon > 0$, there exists a positive integer m such that

$|T_n - T_k| < \varepsilon$ for n > m, k > m.

Let n > k. Now

$S_n = u_1 + u_2 + u_3 + ... + u_k + u_{k+1} + ... + u_n$

$S_n = u_1 + u_2 + u_3 + ... + u_k$

$|S_n - S_k| = |u_{k+1} + u_{k+2} + ... + u_n|$

$\leq |u_{k+1}| + |u_{k+2}| + ... + |u_n|$

$= |T_n - T_k|$

$< \varepsilon$, for n > m, k > m, n > k.

$\therefore$ $\{S_n\}$ is a Cauchy sequence.

$\therefore$ $\sum u_n$ is convergent

Remark : The converse of the above theorem is not true, i.e., if a series is convergent, it may not be absolutely convergent.

Let us consider the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....\infty$$

We know that the series is convergent

But the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....\infty$ is not convergent.

$\therefore$ The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....\infty$ is not absolutely convergent.

Example 7. Let us consider the series

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+.....+\frac{(-1)^{n+1}}{2n+1}+.....\infty$$

Now let us consider the series $\sum|u_n|$, where $u_n=\frac{(-1)^{n+1}}{2n+1}$

$$|u_n|=\left|\frac{(-1)^{n+1}}{2n+1}\right|=\frac{1}{2n+1}$$

Let us consider the series $\sum v_n$, where $\sum v_n=\frac{1}{n}$

$$\sum v_n=\frac{1}{n}$$

Now $\lim_{n\to\infty}\frac{|u_n|}{v_n}=\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{n}{n\left(2+\frac{1}{n}\right)}$

$$=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{1}{2}$$

The series $\sum v_n$ is divergent (Harmonic Series)

$\therefore$ The series $\sum|u_n|$ is also divergent (Theorem 4.5 II, Unit 4).

$\therefore$ The given series is not absolutely convergent.

Again the sequence $\left\{\frac{1}{3},\frac{1}{5},\frac{1}{7},.....\right\}$ is monotonically decreasing.

$$\lim_{n \to \infty} \frac{1}{2n+1} = 0$$

by Leibniz's test, the series $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \ldots\ldots$ is convergent.

Note : We know that if an infinite series is absolutely convergent, then it is convergent. Contrapositively, if a series is not convergent, then it is not absolutely convergent.

Do You Know ?

An infinite series of positive terms cannot be conditionally convergent.

By definition of conditional convergence, $\sum u_n$ converges, but $\sum |u_n|$ diverges. For a series of positive terms, $u_n = |u_n|$.

For a seris of positive terms,

convergence of $\sum u_n$

$\Rightarrow$ convergence of $\sum |u_n|$



CHECK YOUR PROGRESS 1

Q 1: Examine the convergence of the series

i) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots + \frac{x^4}{n^2+1} + \ldots\ldots\infty, \quad x > 0$

ii) $\frac{x}{1} + \frac{1}{2}\frac{x^2}{3} + \frac{1.3}{2.4}.\frac{x^3}{5} + \frac{1.3.5}{2.4.6}.\frac{x^4}{7} + \ldots..\infty, \quad x > 0$

iii) $\sum \frac{1}{n \log n}, \quad n \geq 2$

iv) $1 + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \ldots\ldots\ldots\infty, \quad p > 0$

v) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \ldots\ldots\ldots\infty, \; x > 0$

Q 2. Find the values of p for which the series

$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \ldots\ldots\ldots\infty$ converges conditionally.

Q 3: Show that the series $1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+.....\infty$ is conditionally convergent.

Q 4: Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....\infty$ is convergent, and its sum is log2.

Q 5: Show that the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....\infty$ is convergent, and its sum is $\frac{\pi}{4}$

Do you Know?

If the terms of absolutely convergent series are rearranged, then the series remains convergent and its sum remains unaltered.



5.6 LET US SUM UP

- Eight special tests for determining the convergence of a variey of series have been discussed.
- An infinite series $\sum u_n$ is called an alternating series if any two consecutive terms of the series are of opposite signs.
- Let $\{u_n\}$ be a monotonic decreasing sequence of positive real numbers, and let $\lim_{n\to\infty} u_n = 0$. Then the alternating series $u_1 - u_2 + u_3 - u_4 + ...\infty$ is convergent.
- If the series $\sum_{n=1}^{\infty} |u_n|$ converges, then the series $\sum u_n$ is said to be absolutely convergent.
- If $\sum u_n$ coverges, but $\sum |u_n|$ diverges, then the series $\sum u_n$ is said to be conditionally convergent.
- An absolutely convergent series is convergent.

5.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: We have $u_n = \frac{x^n}{n^2+1}$, (neglecting the first term)

$$\therefore\ u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$\therefore\ \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2+1}.\frac{n^2+1}{x^n}$$

$$= \frac{n^2+1}{(n+1)^2+1}.x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left\{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right\}}.x$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}.x$$

= x

$\therefore$ The given series is convergent if x < 1 and is divergent if x > 1 (by d'Alembert's when x = 1, the series becomes

$$1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+.....+\frac{1}{n^2+1}+.....\infty$$

In this case, $u_n = \frac{1}{n^2+1}$, (neglecting the first term)

Let us consider the series $\sum \upsilon_n$ where $\upsilon_n = \frac{1}{n^2}$

Now $\frac{u_n}{\upsilon_n} = \frac{1}{n^2+1}.\frac{n^2}{1}$

$$\lim_{n\to\infty} \frac{u_n}{\upsilon_n} = \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1+n^2} = 1$$

Now, the series $\sum \frac{1}{n^2}$ is convergent.

$\therefore$ The series $\sum u_n$ is also convergent (by comparison test)

$\therefore$ The given series is convergent if $0 < x \leq 1$ and divergent if x > .

ii) $\frac{x}{1} + \frac{1}{2}\frac{x^2}{3} + \frac{1.3}{2.4}.\frac{x^3}{5} + \frac{1.3.5}{2.4.6}.\frac{x^4}{7} + ...\infty, \; x > 0$

We have

$u_n = \frac{1.3.5.....(2n-1)}{2.4.6.....2n}.\frac{x^{n+1}}{2n+1}.$ (neglecting the first term)

$\therefore u_{n+1} = \frac{1.3.5....(2n-1)\ (2n+1)}{2.4.6.....2n.2(n+1)}.\frac{x^{n+2}}{2(n+1)+1}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{1.3.5.....(2n-1)\ (2n+1)}{2.4.6.....2n.2(n+1)}.\frac{x^{n+2}}{2(n+1)+1}.\frac{2.4.6.....2n}{1.3.5.....(2n-1)}.\frac{2n+1}{x^{n+1}}$

$= \frac{2n+1}{2n+2}.\frac{2n+1}{2n+3}.x$

$\therefore \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{n\ (2+\frac{1}{n})\ n\ (2+\frac{1}{n})}{n\ (2+\frac{2}{n})\ n\ (2+\frac{3}{n})}.x$

= x

$\therefore$ the given series is covergent if x < 1, and is divergent if x > 1 (by d' Alembert's ratio test)

When x = 1, the series becomes

$1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + ...\infty$

Here $u_n = \frac{1.3.5....(2n-1)}{2.4.6.....2n}.\frac{1}{2n+1}$

Now, comparison test is not possible to apply.

In this case, $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{(2n+1)\ (2n+1)}{(2n+2)\ (2n+3)}$

= 1

$\therefore$ d' Alembert's ratio test fails.

Let us now try Raabe's test

Now $n\left(\frac{u_n}{u_{n+1}}-1\right)$

$$=n\left(\frac{(2n+2)\,(2n+3)}{(2n+1)^2}-1\right)$$

$$=n\,\frac{(2n+2)\,(2n+3)-(2n+1)^2}{(2n+1)^2}$$

$$=n\,\frac{4n^2+10n+6-4n^2-4n-1}{(2n+1)^2}$$

$$=\frac{n\,(6n+5)}{(2n+1)^2}$$

$$L=\lim_{n\to\infty}\ n\left(\frac{u_n}{u_{n+1}}-1\right)$$

$$=\lim_{n\to\infty}\ \frac{n\,(6n+5)}{(2n+1)^2}=\lim_{n\to\infty}\ \frac{n\,n\,(6+\frac{5}{n})}{n^2\,(2+\frac{1}{n})^2}$$

$$=\lim_{n\to\infty}\ \frac{(6+\frac{5}{n})}{(2+\frac{1}{n})^2}=\frac{6}{4}=\frac{3}{2}>1$$

$\therefore$ The series is convergent (by Raabe's test)

Thus, the given series is convergent if $0<x\le 1$, and divergent if x > 1.

(iii) The series $\sum_{n=2}^{\infty}\frac{1}{n\log n}$

We will use Cauchy's condensation test.

Here $f(n)=\frac{1}{n\log n}$.

Now { log n} is an increasing sequence.

$\therefore$ log (n + 1) > log n

$\therefore$ (n + 1) log (n + 1) > n log n

$$\therefore \frac{1}{(n+1)\log(n+1)} < \frac{1}{n\log n}$$

$\therefore \{f(n)\}_{n=2}^{\infty}$ is a monotomic decrasing sequence.

We take a = 2

$$\therefore a^n f(a^n) = 2^n f(2^n)$$

$$= 2^n \frac{1}{2^n \log 2^n} = \frac{1}{\log 2^n} = \frac{1}{n\log 2}$$

$\therefore$ The series $\sum f(n)$ is convergent or divergent according as the series $\sum 2^n f(2^n)$, i.e.

$\sum \frac{1}{n\log 2}$ is convergent or divergent.

Now $\frac{1}{\log 2}$ is a constant and the series $\sum \frac{1}{n}$ is divergent. Therefore, the series $\sum \frac{1}{n\log 2}$ is divergent.

Thus, the given series is divergent.

(iv) $1+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+.....\infty, \quad p>0$

Here $u_n = \left(\frac{1.3.5....(2n-1)}{2.4.6.....2n}\right)^p$, (ignoring the first term)

$$u_{n+1} = \left(\frac{1.3.5....(2n-1)\ (2n+1)}{2.4.6.....2n.\ (2n+2)}\right)^p$$

$$\therefore \frac{u_{n+1}}{u_n} = \left(\frac{(2n+1)}{(2n+2)}\right)^p$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{(2+\frac{1}{n})}{(2+\frac{2}{n})}\right)^p = 1$$

$\therefore$ d' Alembert's ratio test fails.

Now, we will apply Logarithmic test.

We have , $\log \frac{u_n}{u_{n+1}}$

$= \log \left(\frac{2n+2}{2n+1}\right)^p$

$= p \log \frac{2n\left(1+\frac{1}{n}\right)}{2n\left(1+\frac{1}{2n}\right)} = p \log \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}$

$= p\left(\log(1+\frac{1}{n}) - \log(1+\frac{1}{2n})\right)$

$= p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}.....\infty\right] - \left(\frac{1}{2n} - \frac{1}{2.4n^2} + \frac{1}{3.8n^3}.....\infty\right)$

$\therefore n \log \frac{u_n}{u_{n+1}}$

$= p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^2}.....\infty\right] - \left(\frac{1}{2} - \frac{1}{8n} + \frac{1}{24n^2}.....\infty\right)$

$\therefore L = \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = p\left(1-\frac{1}{2}\right) = \frac{p}{2}$

$\therefore$ The given series converges if $\frac{p}{2} > 1$ i.e., if p > 2, and diverges if p < 2

when p =2 , then L = 1, and the test fails.

when p =2

Now $\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n$

$= \left[2\left(1 - \frac{1}{2n} - \frac{3}{3n^2}.....\right) - \left(\frac{1}{2} - \frac{1}{8n} + \frac{1}{24n^2}.....\right) - 1\right] \log n$

$= \left[2\left(\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2}.....\right) - 1\right] \log n$

$= = \left(-\frac{3}{4} + \frac{7}{12n}.....\right) \frac{\log n}{n}$

$$\therefore \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n=0<1$$

$\therefore$ The series is divergent if p = 2.

Thus, the series is convergent if p > 2, and divergent if $0<p\le 2$.

(v) Here $u_n=\dfrac{x^n}{n\,(n+1)},\quad x>0$

$$u_{n+1}=\frac{x^n+1}{(n+1)\,(n+2)}$$

$$\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{(n+1)\,(n+2)}\cdot\frac{n(n+1)}{x^n}$$

$$=\frac{n}{n+1}x$$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n}{n+1}x=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}x=x$$

$\therefore$ The given series is convergent when x < 1, and divergent when x > 1. When x = 1, $u_n=\dfrac{n}{n+1}$

Let us consider the series $\sum\upsilon_n$, where $\upsilon_n=\dfrac{1}{n^2}$

Now, $\lim_{n\to\infty}\dfrac{u_n}{\upsilon_n}=\lim_{n\to\infty}\dfrac{1}{n(n+1)}\cdot\dfrac{n^2}{1}=\lim_{n\to\infty}\dfrac{1}{1+\frac{1}{n}}=1$

Again the series $\sum\upsilon_n=\sum\dfrac{1}{n^2}$ is convergent

$\therefore$ The series is convergent for x = 1

Thus, the given series is convergent for $0<x\le 1$, and divergent for x > 1.

Ans to Q No 2: The series is

$$1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+.....\infty$$

Let p > 0

Then $\left\{1,\frac{1}{2^p},\frac{1}{3^p},\frac{1}{4^p}.....\right\}$ is monotonic decreasing sequence.

Again, $\lim_{n \to \infty} \frac{1}{n^p} = 0$

$\therefore$ The series is convergent for p > 0 (by Leibniz's theorem)

Let p = 0. In this case, the series becomes

$1 - 1 + 1 - 1 + \ldots\ldots\ldots\infty$

Now $\lim_{n \to \infty} u_n = \lim_{n \to \infty} (-1)^{n+1}$ does not exist

$\therefore$ The series is divergent for p = 0.

Let p < 0. In this case

$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{(-1)^{n+1}}{n^p}$ does not exist.

$\therefore$ The series is divergent for p < 0.

Thus, we see that the given series is convergent only when p > 0

Again, we know that the series

$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots\ldots\ldots\infty$ is convergent only when p > 1

$\therefore$ The given series is conditionally convergent for $0 < x \leq 1$

Ans to Q No 3: The series is

$1 - \frac{1}{4}, + \frac{1}{7}, \frac{1}{10} + \ldots\ldots\ldots\ldots$

clearly $\left\{1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \ldots\ldots\ldots\ldots \frac{1}{3n-2}, \ldots\ldots\ldots\ldots\infty\right\}$ is monotonic decreasing sequence.

$\lim_{n \to \infty} \frac{1}{3n-2} = 0$

$\therefore$ The series is convergent.

Now let us consider the series

$1 + \frac{1}{4} + \frac{1}{7} + \ldots\ldots\ldots\ldots + \frac{1}{3n-2} + \ldots\ldots\infty$

Here $u_n = \frac{1}{3n-2}$

We take the series $\sum \upsilon_n$, where $\upsilon_n = \frac{1}{n}$

Now $\lim_{n\to\infty}\frac{u_n}{\upsilon_n}=\lim_{n\to\infty}\frac{1}{3n-2}.\frac{n}{1}=\lim_{n\to\infty}\frac{n}{n(3-\frac{2}{n})}$

$$=\lim_{n\to\infty}\frac{1}{3-\frac{2}{n}}=\frac{1}{3}\neq 0$$

We know that the series $\sum\upsilon_n$ is divergent.

$\therefore$ by comparison test, the series $\sum u_n$ is also divergent.

$\therefore$ The given series is conditionally convergent.

Ans to Q No 4: We have already proved that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....\infty$ is convergent.

Now $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}.....\infty, \quad -1<x\leq 1$

$\therefore 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....\infty=\log 2$ (putting x = 1)

Ans to Q No 5: $\left\{1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\ .....\right\}$ is a monotonic decreasing sequence.

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{2n-1}=0$$

$\therefore$ The series is convergent. (by Leibniz's test)

Now $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}.....\infty, \ -1\leq x\leq 1$

$\therefore 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....\infty=\tan^{-1}1=\frac{\pi}{4}$



5.8 FURTHER READING

1) Fraleigh, John B : A first course in abstract Algebra, Pearson Education (Singapure) Pvt. Ltd. India Branch, Delhi - 11092.

2) Singh, S and Qazi, Z : Modern Algebra, Vikas Publicasting House Pvt. Ltd, New Delhi - 110014.

3) Macauley, M. : Section 6 : Fields and Galois Theory, http :// www.math.clemson.edu/~macaule/.

4) Talati, A. : Class Notes of Professor June Huh, 2022.

5) Singh, J. : Theory of Field Extension, Maharshi Dayanand University Press, Rohtal-124001.



5.9 MODEL QUESTIONS

Q 1: Examine the convergence of the following series

(i) $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+.....+\frac{n^2-1}{n^2+1}x^n+.....\infty,\quad x>0$

(ii) $\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+.....\infty,\quad x>0$

(iii) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+.....\infty,\quad x>0$

(iv) $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+.....\infty,\quad x>0$

(v) $\frac{1}{2}+\frac{1}{2+1}+\frac{1}{2^2+1}+\frac{1}{2^3+1}+.....\infty$

(vi) $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+.....\infty,\quad x>0$

Q 2: Discuss the convergence of the series

(i) $\sum u_n$, where $u_n=\frac{1}{\log n},\quad n\geq 2$

(ii) $\sum u_n$, where $u_n=\frac{1}{n(\log n)^p},\quad n\geq 2,\ p>0$

Q.3: Examnine the convergence of the series

$$\frac{1^2}{2^2}+\frac{1^2}{2^2}.\frac{3^2}{4^2}x+\frac{1^2}{2^2}.\frac{3^2}{4^2}.\frac{5^2}{6^2}x^2+.....\infty,\quad x>0$$

[Here $u_n=\frac{1^2.3^2.5^2.....(2n-1)^2}{2^2.4^2.6^2.....(2n)^2}\ x^{n-1}$

Applying d'Alembert's ratio test, we wil find that the series is convergent for x < 1, and divergent for x > 1.

For x = 1, Raabe's test fails. Applying De Morgan and Bertrand's test, we will find that the series is divergent for x = 1]

Q 4: Test the convergence of the following series

(i) $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+.....\infty$

(ii) $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+.....\infty$

Q 5: Examine if the series $1-\frac{1}{2}+\frac{1.3}{2.4}-\frac{1.3.5}{2.4.6}+.....\infty$ is conditionally convergent.

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